The Solution of the Riemann Hypothesis.

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ABSTRACT
This work is dedicated to the promotion of the results Abels obtained modifying zeta functions. The properties of zeta functions are studied; these properties lead to new regularities of zeta functions. The choice of a special type of modified zeta functions allows estimating the Riemanns zeta function and solving Riemann Problem- Millennium Prize Problems.

INTRODUCTION

In this work we are studying the properties of modified zeta functions. Riemann's zeta function is defined by the Dirichlet's distribution

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it \tag{1}$$

absolutely and uniformly converging in any finite region of the complex s-plane, for which $\sigma \geq 1 + \epsilon, \epsilon > 0$. If $\sigma > 1$ the function is represented by the following Euler product formula

$$\varsigma(s) = \prod_p \left[1 - \frac{1}{p^s}\right]^{-1} \tag{2}$$

where p is all prime numbers. $\varsigma(s)$ was firstly introduced by Euler 1737 in [1], who decomposed it to the Euler product formula (2). Dirichlet and Chebyshev, studying the law of prime numbers distribution, had considered this in [2] . However, the most profound properties of the function $\varsigma(z)$ had only been discovered later, when the function had been considered as a function of a complex variable. In 1876 Riemann was the first who in [3] that :
$\varsigma(s)$ allows analytical continuation on the whole z-plane in the following form

$$\pi^{-s/2}\Gamma(s/2)\varsigma(s) = 1/(s(s-1)) + \int_1^{+\infty} (x^{s/2} - 1 + x^{(1-s)/2} - 1)\theta(x)dx. \tag{3}$$

where $\Gamma(z)$- gamma function,
$\theta(x) = \sum_{n=1}^{\infty} exp(-\pi n^2 x)$.

$\varsigma(s)$ is a regular function for all values of s, except s=1, where it has a simple pole with a deduction equal to 1, and satisfies the following functional equation

$$\pi^{-s/2}\Gamma(s/2)\varsigma(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\varsigma(1-s) \tag{4}$$

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This equation is called the Riemann's functional equation.
The Riemann's zeta function is the most important subject of study and has a plenty of interesting generalizations. The role of zeta functions in the Number Theory is very significant, and is connected to various fundamental functions in the Number Theory as Mobius function, Liouville function, the function of quantity of number divisors, the fucntion of quantity of prime number divisors. The detailed theory of zeta functions is showed in [4]. The zeta function spreads to various disciplines and now the function is mostly applied in quantum statistical mechanics and quantum theory of pole[5-7]. Riemann's zeta function is often introduced in the formulas of quantum statistics. A well-known example is the Stefan-Boltzman law of a black body's radiation. The given aspects of the zeta function reveal global necessity of its further investigation.
We are interested in the analytical properties of the following generalizations of zeta functions:

$$P(s) = \sum_{p} \frac{1}{p^s},\ Q(s) = \sum_{p} \frac{1}{(p-1)^s} \tag{5}$$

$$P_m(s) = \sum_{p\leq m} \frac{1}{p^s},\ Q_m(s) = \sum_{p\leq m} \frac{1}{(p-1)^s}, \tag{6}$$

$$P^m(s) = \sum_{p>m} \frac{1}{p^s},\ Q^m(s) = \sum_{p>m} \frac{1}{(p-1)^s} \tag{7}$$

$$\varsigma_p^m(s) = \varsigma(s) - P^m(s) \tag{8}$$

$$\varsigma^m(s) = \sum_{n=1}^{m} \frac{1}{n^s},\ \ s = \sigma + it \tag{9}$$

where p are prime numbers. The forms of the given function (7)-(8) allow to assume that they possess the same properties as the zeta function (1), but it is not quite obvious, considering

$$ln(\varsigma(s)) = \sum_{n=1}^{\infty} P(ns)/n. \tag{10}$$

$$|R2(s)| = |\sum_{m=2}^{\infty} P(ms)/m| \leq \sum_{m=2}^{\infty} |P(ms)/m| \leq C_\epsilon \sum_{m=2}^{\infty} |-2^{m\epsilon}/m| < CC_\epsilon < \infty \tag{11}$$

we see the necessity of analyzing (5)-(8) functions for a deeper understanding of the properties of zeta functions.

RESULTS

These are the well-known Abel's results.

Theorem 1 *Let the function ϕ be limited on every finite interval, and $\frac{d\phi}{dx}$ is continuous and limited on every finite interval then*

$$\sum_{a<n\leq b} \phi(n) = \int_a^b \phi(x)dx + \int_a^b (x - [x] - 1/2)\frac{d\phi}{dx}dx + (a - [a] - 1/2)\phi(a) - (b - [b] - 1/2)\phi(b) \tag{12}$$

Corollary 1 *Let the function $\phi(x) = x^{-s}$, $s > 1, a, b \in N$ then*

$$\sum_{a<n\leq b} n^{-s} = \frac{b^{1-s} - a^{1-s}}{1-s} - s\int_a^b \frac{(x - [x] - 1/2)}{x^{s+1}}dx + \frac{1}{2}(b^{-s} - a^s) \tag{13}$$

Corollary 2 *Let the function* $\phi(x) = x^{-s}$, $s > 1, a = 1, b = \infty$ *then*

$$\sum_{1<n<\infty} n^{-s} = -\frac{1}{1-s} - s\int_1^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx - \frac{1}{2} \tag{14}$$

Let N be the set of all natural numbers and $N_p^m = (n \in N, n \geq m, n - prime\, number)$ $NP_m = N \setminus N_p^m$ - the set of all natural numbers without N_p^m

Below we will always let $m > 3$, this limitation is introduced only to simplify the calculations. Considering all the information above let us rewrite

$$\varsigma_p^m(s) = \sum_{n\in NP_m} \frac{1}{n^s}.$$

To use the results of Abel, we introduce the set

$AB_p^m = \left\{< a, b > |a \in N_p^m, b \in N_p^m, b > a \geq m, (a,b) \bigcap N_p^m = \emptyset\right\}$.

For the function $P^m(s)$, let us apply the results obtained by Abel's for the zeta function representation. For simplicity, we formulate several lemmas.

Lemma 1 *Let the function*

$$\delta(s) = P^m(s) - Q^m(s), then \tag{15}$$

$$\delta(s) = -sP^m(s+1) + s^2 O(P^m(s+2)). \tag{16}$$

PROOF: According to the theorem conditions we have

$$\delta(s) = \sum_{p\in N_p^m}\left[\frac{1}{p^s} - \frac{1}{(p-1)^s}\right] = \sum_{p\in N_p^m}\frac{1}{p^s}\left[1 - \frac{1}{(-1/p+1)^s}\right] = \tag{17}$$

$$= -s\sum_{p\in N_p^m}\frac{1}{p^{s+1}} + s^2 O(P^m(s+2)). \tag{18}$$

Lemma 2 *Let the function*

$$\gamma 1(s) = \sum_{p\in N_p^m}\int_{p-1}^{p}\frac{x}{x^{s+1}}dx,\ \gamma 2(s) = -\sum_{p\in N_p^m}\int_{p-1}^{p}\frac{[x]}{x^{s+1}}dx,\ \gamma 3(s) = -\sum_{p\in N_p^m}\int_{p-1}^{p}\frac{1/2}{x^{s+1}}dx, \tag{19}$$

$$then, \tag{20}$$

$$\gamma 1(s) = \frac{1}{1-s}\sum_{p\in N_p^m}\left[\frac{1}{p^{s-1}} - \frac{1}{(p-1)^{s-1}}\right] = \frac{\delta(s-1)}{1-s} \tag{21}$$

$$\gamma 2(s) = -\frac{1}{s}\sum_{p\in N_p^m}\left[\frac{p-1}{p^s} - \frac{p-1}{(p-1)^s}\right] = -\frac{\delta(s-1)}{s} + \frac{P^m(s)}{s} \tag{22}$$

$$\gamma 3(s) = -\frac{1}{2s}\sum_{p\in N_p^m}\left[\frac{1}{p^s} - \frac{1}{(p-1)^s}\right] = -\frac{\delta(s)}{2s}. \tag{23}$$

$$s\left[\gamma 1(s) + \gamma 2(s) + \gamma 3(s)\right] = s\left[\frac{\delta(s-1)}{s-1} - \frac{\delta(s-1)}{s} + \frac{P^m(s)}{s} - \frac{\delta(s)}{2s}\right] \tag{24}$$

PROOF: Follows from computing of integrals.

Lemma 3 *Let the function*
$\phi(x) = x^{-s}, s > 1$, *then*

$$-\delta(s-1) - m^{1-s} = \sum_{<a,b>\in AB_p^m}\left[(b-1)^{1-s} - a^{1-s}\right], \tag{25}$$

$$\sum_{<a,b>\in AB_p^m} s\int_a^{b-1}\frac{(x-[x]-1/2)}{x^{s+1}}dx = s\int_m^\infty\frac{(x-[x]-1/2)}{x^{s+1}}dx - s\left[\gamma 1(s) + \gamma 2(s) + \gamma 3(s)\right]; \tag{26}$$

PROOF: Computing the sums , we have

$$\sum_{<a,b>\in AB_p^m} \left[(b-1)^{1-s} - a^{1-s}\right] = -m^{1-s} + \sum_{p\in N_p^m} \left[(p-1)^{1-s} - p^{1-s}\right] = -m^{1-s} - \delta(s-1) \tag{27}$$

The second assertion follows from the properties of the integral.

Theorem 2 *Let the function*
$\phi(x) = x^{-s}, s > 1$ *, then*

$$sP^m(s) = \varsigma(s) + m^{1-s} + s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx + \delta(s) - m^{-s} - O(P^m(s+1))) \tag{28}$$

PROOF: Using orollary 1. we have

$$\varsigma_p^m(s) = \varsigma^m(s) + \sum_{<a,b>\in AB_p^m}\sum_{a<n<b} n^{-s} = \varsigma^m(s) + \sum_{<a,b>\in AB_p^m} \frac{(b-1)^{1-s} - a^{1-s}}{1-s} - \tag{29}$$

$$\sum_{<a,b>\in AB_p^m} s\int_a^{b-1} \frac{(x-[x]-1/2)}{x^{s+1}}dx + \frac{1}{2}\sum_{<a,b>\in AB_p^m}((b-1)^{-s} - a^{-s}). \tag{30}$$

Using Lemma 1-3 we have

$$\varsigma_p^m(s) = \varsigma^m(s) - \frac{1}{1-s}\left(\delta(s-1) + m^{1-s}\right) \tag{31}$$

$$-s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx + s\left[\gamma 1(s) + \gamma 2(s) + \gamma 3(s)\right] - \frac{1}{2}\left(\delta(s) + m^{-s}\right) \tag{32}$$

Using Lemma 3 we have

$$\varsigma_p^m(s) = \varsigma^m(s) - \frac{1}{1-s}\left(\delta(s-1) + m^{1-s}\right) - s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx + \tag{33}$$

$$s\left[\frac{\delta(s-1)}{1-s} - \frac{\delta(s-1)}{s} + \frac{P^m(s)}{s} - \frac{\delta(s)}{2s}\right] - \frac{1}{2}\left(\delta(s) + m^{-s}\right) \tag{34}$$

Simplifying the last expression, we obtain

$$\varsigma_p^m(s) = \varsigma^m(s) + \delta(s-1)\left[\frac{s-1}{1-s} - 1\right] + P^m(s) - \frac{m^{1-s}}{1-s} - s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx - \tag{35}$$

$$\frac{1}{2}\left(\delta(s) + m^{-s}\right) \tag{36}$$

$$\varsigma(s) - P^m(s) = \varsigma^m(s) - 2\delta(s-1) + P^m(s) - \frac{m^{1-s}}{1-s} - s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx - \tag{37}$$

$$\frac{1}{2}\left(\delta(s) + m^{-s}\right) + O(P^m(s+1)). \tag{38}$$

Using Lemma 1 we have

$$\varsigma(s) = \varsigma^m(s) + 2sP^m(s) - \frac{m^{1-s}}{1-s} - s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx - \frac{1}{2}\left(\delta(s) + m^{-s}\right) + O(P^m(s+1)). \tag{39}$$

Computing $sP^m(s)$, we obtain

$$2sP^m(s) = \varsigma(s) - \varsigma^m(s) + \frac{m^{1-s}}{1-s} + s\int_m^\infty \frac{(x-[x]-1/2)}{x^{s+1}}dx + \frac{1}{2}\left(\delta(s) + m^{-s}) - O(P^m(s+1))\right) \tag{40}$$

From the last equation we obtain the regularity of the function $P^m(s)$ as $1/2 < Re(s) < 1$.

Theorem 3 *The Riemann's function has nontrivial zeros only on the line Re(s)=1/2;*
PROOF: For $R2(s) = \sum_{m=2}^\infty P(ms)/m$, we have

$$|R2(s)| = |\sum_{m=2}^\infty P(ms)/m| \leq \sum_{m=2}^\infty |P(ms)/m| \leq C_\epsilon \sum_{m=2}^\infty |-2^{m\epsilon}/m| < CC_\epsilon < \infty \tag{41}$$

Applying the formula from the theorem 2

$$ln(\varsigma(s)) = P(s) + \sum_{m=2}^\infty P(ms)/m = P(s) + R2(s) = P_m(s) + P^m(s) + R2(s) \tag{42}$$

estimating by the module

$$|ln(\varsigma(s)| \leq |P^m(s)| + |R2(s)| + |P_m(s)|\,. \tag{43}$$

Estimating the zeta function, potentiating, we obtain

$$|(\varsigma(s)| \geq \exp\left[-|P^m(s)| - |R2(s)| - |P_m(s)|\right] \tag{44}$$

According to Theorem 2, $|P^m(s)|$ limited for s from the following set

$$(s, |s| < R, |s| > 1 + \epsilon, \epsilon > 0) \tag{45}$$

finally we obtain:

$$|(\varsigma(s)| \geq \exp\left[-C_R\right],\ Re(s) > 1/2 + 1/R, |s| < R, |s| > 1 + \epsilon, \epsilon > 0 \tag{46}$$

These estimations for $|P(s)|, |R2(s)|, |P_m(s)|$ prove that function does not have zeros on the half-plane Re(s)¿1/2 +1/R due to the integral representation (3) these results are projected on the half-plane Re(s)¡1/2 . The Riemann's hypothesis is proved.

CONCLUSION

In this work we obtained the estimation of the Riemann's zeta function logarithm outside of the line Re(s)=1/2 and outside of the pole s=1. This work accomplishes all the works of the greatest mathematicians, applying their immense achievements in this field. Without their effort we could not even attempt to solve the problem.

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